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2000 J. Phys.: Condens. Matter 12 8065

(http://iopscience.iop.org/0953-8984/12/37/305)

Stability of incompressible helium II: a two-fluid system

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Received 28 January 2000, in final form 4 July 2000

Abstract. A numerical attempt to achieve stability of the two-fluid system incompressible helium II is presented. The Orr–Sommerfeld equation for the planar Poiseuille flow of normal fluid with relaxed velocity slip along the interface with a superfluid is solved. The numerical approach is a modified version (with complex matrix pre-conditioning) of a spectral method. The relaxed results are compared with non-relaxed ones which are subject to no-slip boundary conditions. The relaxed slip-flow effect becomes more dominant as the interface parameter increases, because then the critical Reynolds number (or the critical velocity) decreases.

1. Introduction

A most striking characteristic of liquid helium is that it exists in the liquid state down to absolute zero temperature because: (a) the van der Waals forces in helium are weak; (b) the zero-point energy, due to the light mass, is large. In fact, it is called a quantum liquid due to these kinds of quantum effect, which are closely related to the Bose condensation for He II. The well-known properties of He II can be largely accounted for on the basis of phenomenological two-fluid theory [1, 2]. One of the basic assumptions of the two-fluid model is that He II consists of a kind of mixture of two components, a normal component and a superfluid component. The former has viscosity while the superfluid can move without friction as long as certain velocity limits are not exceeded.

One crucial issue for the related research concerning He II is the critical velocity (when there is flow through a capillary or plane channel), which depends on the micro-channel size. Landau attributed the existence of a critical velocity in He II to the breakdown of the superfluid due to the creation of excitations (he proposed that phonons and rotons are two types of excitation which make up the normal fluid). If the velocity is less than the critical value, there will be no dissipation or friction along the boundary or interface for the flow. Note that if the helium is heated (e.g. by the viscous-dissipation effect), then it undergoes (phase) transitions to excited states.

We assume that the critical velocity can be qualitatively investigated by studying its macroscopic or hydrodynamic stability characteristics.

The traditional starting point of an investigation of hydrodynamic stability is eigenvalue analysis, which proceeds as follows: (i) linearize the laminar solution and then (ii) look for unstable eigenvalues of the linearized problem. In much of the literature on linear hydrodynamic stability [3], attention has been restricted to 2D perturbations in view of *Squire's theorem*; in particular, the well-known Orr–Sommerfeld equation is an eigenvalue equation for 2D perturbations. For planar Poiseuille flow, with no-slip boundary conditions (which

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0953-8984/00/378065+05$30.00

are normally valid for macro-channels) [4], eigenvalue analysis predicts a critical Reynolds number $Re = 5772$ [5] as that at which instability should first occur; but in the laboratory, transition to turbulence is observed for Re as low as ~ 1000 [6].

We assume that, in a two-fluid model for incompressible helium II [7], one effect may play a particularly significant role: velocity slip [8–11] along the interface between the normal fluid and superfluid [8, 9] (please see reference [8], especially for slip flow of quantum fluids). The distinction among various flow regimes can be established by introducing the Knudsen number (which also characterizes the value of the velocity slip [8, 10, 11]), $K_n = \lambda/L$, where λ is the mean free path and L is the characteristic flow dimension [8, 10, 11]. Slip-flow conditions exist for $0.001 < K_n \leqslant 0.1$, where the flow can be considered continuous if the velocity slip at the walls is taken into account [8, 10, 11]. This is due to the incomplete momentum and energy exchange between the fluid molecules and the solid boundaries [8–11].

Frisch *et al* [12] recently derived incompressible Navier–Stokes equations which are verified models for dissipated or viscous fluids, from microscopic Boltzmann models for Knudsen number much smaller than one (see also [13]). In the present work, the stability of incompressible slip flow in a normal fluid bounded by two layers of superfluid is compared with the traditional fluid flow in a macro-channel. That is to say, we are considering the flow of ^{3}He in a channel covered with a superfluid ^{4}He film. That is, we will relax the no-slip boundary conditions which are frequently used for traditional macro-channels to the slip boundary conditions, which should be taken into account in this work when we consider the slip flow [8]. The verified code developed by Chu and Chang [14] will be extended here to include relaxed boundary conditions along the interfaces to obtain the stability characteristics of the slip flow.

2. Governing equations

Macroscopically, the motion of the fluid (helium II) as a whole for the two-fluid (normal-fluid and superfluid) system can be treated independently for different parts by using hydrodynamical models (for dissipated and ideal fluid, respectively) starting from the microscopic atomic wave function [1, 2]. The non-dimensional equations of motion for an incompressible normal-fluid flow [3, 7], in the absence of body forces and moments, reduce to

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{U} = -\nabla P + \frac{1}{Re}\nabla^2\boldsymbol{U} \tag{1}$$

where $Re = \rho\bar{U}h/\mu$ is the Reynolds number. For the case of normal-fluid flow driven by a constant pressure gradient, i.e., planar Poiseuille flow, the length scale is the halfwidth of the normal-fluid layer h, and the velocity scale is the centreline velocity $\bar{U}$. Following the usual assumptions of linearized stability theory, $U_i(x_i, t) = \bar{u}_i(x_i) + u_i'(x_i, t)$ and, similarly, $P(x_i, t) = \bar{p}(x_i) + p'(x_i, t)$, the linearized equation which governs the disturbances is

$$\frac{\partial u_i'}{\partial t} + (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})u_i' + (\boldsymbol{u}'\cdot\boldsymbol{\nabla})\bar{u}_i = -\nabla p' + \frac{1}{Re}\nabla^2 u_i'. \tag{2}$$

Disregarding the lateral disturbances, $w' = 0$, a stream function for the disturbance, ψ, can be defined such that $u' = \partial\psi/\partial y$, $v' = -\partial\psi/\partial x$. Using normal-mode decomposition analysis, ψ may be assumed to have the form $\psi(x, y, t) = \phi(y)\exp[\mathrm{i}\alpha(x - ct)]$. Substituting the stream function and eliminating the pressure, we have the linearized disturbance equation

$$(D^2 - \alpha^2)(D^2 - \alpha^2)\phi = \mathrm{i}\alpha\, Re[(\bar{u} - c)(D^2 - \alpha^2)\phi - (D^2\bar{u})\phi] \tag{3}$$

where $D = \mathrm{d}/\mathrm{d}y$. This is also valid for the slip-flow regime, $0.001 < K_n \leqslant 0.1$, since the flow can still be considered as continuous.

2.1. Boundary conditions for the basic flow

For the slip flow, the continuous models can be used if the no-slip boundary condition is modified. A few models have been suggested for estimating the non-zero velocity at a solid surface [7–11]. In this study, we have adopted the approach based on Taylor's expansion of the velocity around the wall (cf. page 34 or 41 in reference [8]). Thus, the first-order approximation yields $\bar{u}|_{wall} = K_n \, \mathrm{d}\bar{u}/\mathrm{d}y$. Consequently, the mean (basic) velocity profile is given by

$$\bar{u} = 1 - y^2 + 2K_n \tag{4}$$

for $-1 \leqslant y \leqslant 1$. The boundary conditions for ϕ or $D\phi$ are the same as in the previous no-slip approach, i.e., $\phi(\pm 1) = D\phi(\pm 1) = 0$.

2.2. Numerical approach

The eigenvalue problem derived above was solved by using the verified code [14], which used a spectral method [15] based on the Chebyshev-polynomial-expansion approach, after the equation and boundary conditions were discretized. The algebraic equation is

$$\begin{aligned}
&\frac{1}{24} \sum_{\substack{p=n+4 \\ p\equiv n\,(\mathrm{mod}\,2)}}^{N} [p^3(p^2-4)^2 - 3n^2p^5 + 3n^4p^5 + 3n^4p^3 - pn^2(n^2-4)^2]a_p \\
&\quad - \sum_{\substack{p=n+2 \\ p\equiv n\,(\mathrm{mod}\,2)}}^{N} \left\{ \left[2\alpha^2 + \frac{1}{4}\,\mathrm{i}\alpha\, Re(4f - 4\lambda - c_n - c_{n-1}) \right] p(p^2-n^2) \right. \\
&\quad \left. - \frac{1}{4}\,\mathrm{i}\alpha\, Re\, c_n p[p^2-(n+2)^2] - \frac{1}{4}\,\mathrm{i}\alpha\, Re\, d_{n-2} p[p^2-(n-2)^2] \right\} a_p \\
&\quad + \mathrm{i}\alpha\, Re\, n(n-1)a_n + \{\alpha^4 + \mathrm{i}\alpha\, Re\,[(f-\lambda)\alpha^2 - 2]\}c_n a_n \\
&\quad - \frac{1}{4}\,\mathrm{i}\alpha^3\, Re\,[c_{n-2}a_{n-2} + c_n(c_n + c_{n-1})a_n + c_n a_{n+2}] = 0
\end{aligned} \tag{5}$$

for $n \geqslant 0$, $f = 1 + 2K_n$, where $c_n = 0$ if $n > 0$, and $d_n = 0$ if $n < 0$, $d_n = 1$ if $n \geqslant 0$. The boundary conditions become

$$\sum_{\substack{n=0 \\ n\equiv 0\,(\mathrm{mod}\,2)}}^{N} a_n = 0 \qquad \sum_{\substack{n=0 \\ n\equiv 0\,(\mathrm{mod}\,2)}}^{N} n^2 a_n = 0 \tag{6}$$

$$\sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\,2)}}^{N} a_n = 0 \qquad \sum_{\substack{n=1 \\ n\equiv 1\,(\mathrm{mod}\,2)}}^{N} n^2 a_n = 0. \tag{7}$$

The matrices thus formed are ill conditioned because they are not diagonal or symmetric. So before we perform floating-point computations to get the complex eigenvalues, we should precondition these complex matrices to reduce the errors. Here we adapt Osborne's algorithm [16] to precondition these complex matrices via rescaling, i.e., by performing certain diagonal similarity transformations of the matrix (the errors are in terms of the Euclidean norm of the matrix) designed to reduce its norm. The details of this algorithm are given in [17]. The form of the reduced matrix is upper Hessenberg; then we perform the stabilized LR transformations for these matrices to get the eigenvalues [18] (please see also [23] for the details). Preliminary verified results from this numerical code [14] have been obtained for the cases of $K_n = 0$ (no-slip boundary conditions) and compared with the benchmark results of Orszag [5] and those for

2D cases of Li and Widnall [19]. For example, for $Re = 10\,000.0$, $\alpha = 1.0$ for the test case: planar Poiseuille flow [5], we obtained the same spectrum as for 0.237 526 48 + i 0.003 739 67 for $Cr + \mathrm{i}\, Ci$ [14] which Orszag obtained from CDC 7600 in 1971.

3. Results and discussion

Figure 1 compares the neutral curves for $K_n = 0$, 0.001, and 0.01. It is clear that as K_n increases, from 0 to 0.001, and 0.01, the critical Reynolds number Re_c decreases, from 5772 to 4836, and 1206 respectively. Hence, the velocity slip at the wall degrades the stability of the flow. These results show a similar trend to those reported in [20]. Spectra of the Orr–Sommerfeld operator were also calculated for the slip and no-slip flow ($K_n = 0$) at $\alpha = 1$ for Reynolds numbers ($Re = 400$, 600, 1000, 1500) selected on the basis of the previous observations mentioned in section 1 and references [6, 21, 22]. The shift in the spectra is significant as K_n increases to 0.001 for all the Reynolds numbers [23]. This indicates that in the slip-flow regime the rarefaction or velocity-slip effect becomes dominant. Thus, as the flow scale decreases, slip-flow effects result in a smaller Re_c or critical velocity. The excitations for He II have an earlier onset for smaller micro-channel size and larger velocity slip. An early transition to superfluid turbulence (due to smaller critical velocity) might result in more viscous dissipations (or heatings) which will then trigger earlier phase transition. This result is related to the issue of when the normal-fluid profile becomes unstable. The stream function

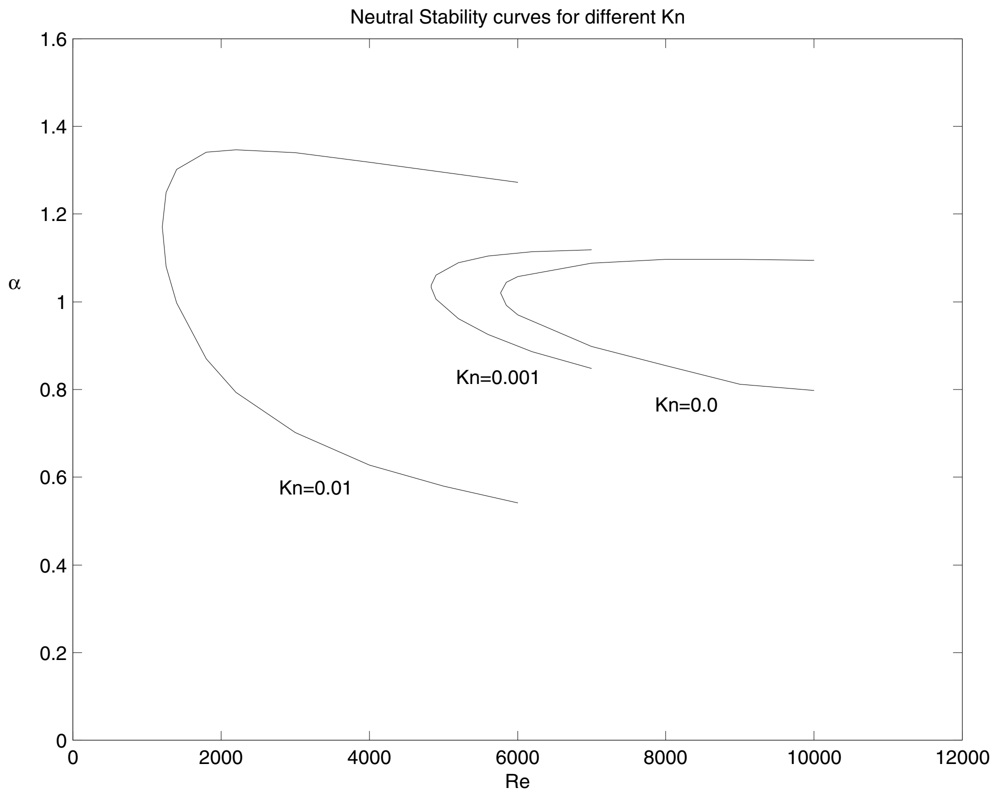

Figure 1. Velocity-slip effects (K_n) on the neutral stability boundary of planar Poiseuille flow for the normal fluid.

ψ of our formulation could be related to the vorticity in hydrodynamics [3, 7, 9]; thus, the unstable mode of ψ might give clues useful to the study of vortex perturbations in normal fluid. It cannot, however, give us more clues useful to the investigation of this issue: the generation of vortex lines when the superfluid exceeds a critical velocity [8, 24].

To conclude briefly, the (incompressible) Orr–Sommerfeld equation was numerically solved to study the stability of the planar Poiseuille flow in incompressible helium II for the normal-fluid part. The relaxed velocity slip along the interface between superfluid and normal fluid decreases the critical Reynolds number or the critical velocity (for the same geometric scale and properties of the fluid). Moreover, a slip-flow effect may account for some of the discrepancy between the theoretical and experimental data [24], since the present results are much closer to the reported experimental values [6, 21, 22, 24].

Acknowledgments

The author would like to thank the referees for their valuable comments and suggestions. The author is also indebted to Jie Q-Y (PO Box 30-15, Shanghai 200030).

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